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On boson algebras as Hopf algebras

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Abstract. Certain types of generalized undeformed and deformed boson algebras which admit a Hopf algebra structure are introduced, together with their Fock-type representations and their corresponding R-matrices. It is also shown that a class of generalized Heisenberg algebras including those underlying physical models such as that of Calogero–Sutherland, is isomorphic with one of the types of boson algebra proposed, and can be formulated as a Hopf algebra.

1. Introduction

Deformations of the boson algebra have recently been the subject of extensive research partly because of their significance in quantum groups (see for example [1–5]) and supergroups [6]. Chronologically first comes the Arik–Coon q-deformation of the Heisenberg algebra [7]:

$$aa^\dagger - qa^\dagger a = I \tag{1}$$

followed by the Macfarlane–Biedenharn [8, 9], and Sun and Fu [10] q-deformed bosons

$$aa^\dagger - q^{\pm 1}a^\dagger a = q^{\mp N}. \tag{2}$$

The Chakrabarti–Jaganathan two-parameter model [11]

$$aa^\dagger - pa^\dagger a = q^{-N} \tag{3}$$

and the Calogero–Vasiliev model [12]

$$[a, a^\dagger] = I + 2\nu K \qquad K^2 = I \tag{4}$$

which coupled with (2) as

$$aa^\dagger - qa^\dagger a = q^{-N}(I + 2\nu K) \tag{5}$$

was studied in [13], while its q-deformation by Macfarlane [14] is

$$aa^\dagger - q^{\pm(I+2\nu K)}a^\dagger a = [I + 2\nu K]_q q^{\mp(N+\nu-\nu K)} \qquad K = (-1)^N \tag{6}$$

where as usual $[x]_q = (q^x - q^{-x})/(q - q^{-1})$. In addition, with the Katriel–Quesne minimally deformed oscillators [28] which provides an attempt to unify existing deformed oscillators, generalizations and applications of the above models to mathematics and physics have been of increasing interest, and their consistency, interrelation and representations have been well analysed [15–27].

Generalizations of the usual Heisenberg algebra that have appeared in [12] have been implemented [29, 30], in describing relativistic fields with arbitrary fractional spin (anyon),

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'bosonizing' supersymmetric quantum mechanics and pointing out the relation of it with integrable quantum mechanical models [31] such as the Calogero–Sutherland model [32, 33].

On the other hand, the recent investigation [34–38] of simplest q-deformations of the Heisenberg algebra has also been shown to play a key part in obtaining and classifying representations of deformed boson algebras [39].

The relation of a possible Hopf algebra structure consistent both with an appropriate definition of a boson algebra and its deformation have also been addressed [23, 40–43]. In particular in [41–43] a certain definition of deformed boson algebra was investigated having a Hopf algebra structure, while in [44] the R-matrix obtained was corrected and generalized using the quantum double construction. The results of [41] were partly generalized in [23].

The aim of this paper is to investigate certain generalizations of undeformed and deformed boson algebras possessing a Hopf algebra structure, which in [45] will be used to establish an algebraic relation with already known boson algebras (undeformed and deformed). In section 2, after introducing general notions on quasitriangular Hopf algebras, we present and analyse the properties of undeformed generalized boson algebras, $B_\zeta(\alpha, \beta)$, $\zeta = \pm 1$, which admit a Hopf algebra structure, while in section 3 we give a q-deformation $B_\zeta^q(\alpha, \beta)$ of the previous algebras, prove that they also admit a Hopf structure and present an R-matrix for them. We further analyse, in section 4, a more general form of the 'deformed' Heisenberg algebra H_ν of [12], showing that under certain conditions it admits a Hopf algebra structure and demonstrate its connection with the undeformed boson algebra $B_\zeta(\alpha, \beta)$ defined in the second section. Finally, in section 5 we end with certain comments on possible physical and mathematical applications and consequences of our approach.

2. The undeformed generalized boson algebras $B_\zeta(\alpha, \beta)$

We begin by stating certain generalities on quasitriangular Hopf algebras needed later. Consider a unital associative algebra, over a field F, with multiplication $m : A \otimes A \to A$ (i.e. $m(a \otimes b) = ab$, $\forall a$ and $\forall b \in A$) and unit $u : F \to A$ (i.e. $u(1) = I$, the identity on A) endowed with a Hopf algebra structure (cf [47]), that is, having a coproduct $\Delta : A \to A \otimes A$, a counit $\varepsilon : A \to F$ (which is a homomorphism) and an antipode $S : A \to A$ (which is an antihomomorphism, i.e. $S(ab) = S(b)S(a)$, and we shall assume that it has an inverse S^{-1}) subject to the following consistency condition:

$$\begin{aligned} &(\mathrm{id} \otimes \Delta)\Delta(a) = (\Delta \otimes \mathrm{id})\Delta(a) \\ &(\mathrm{id} \otimes \varepsilon)\Delta(a) = (\varepsilon \otimes \mathrm{id})\Delta(a) = a \\ &m(\mathrm{id} \otimes S)\Delta(a) = m(S \otimes \mathrm{id})\Delta(a) = \varepsilon(a)I \qquad \forall a \in A. \end{aligned} \tag{7}$$

Let T be the twist map on $A \otimes A$ defined by $T(a \otimes b) = b \otimes a$. Then an opposite Hopf algebra structure also exists on A with coproduct $T\Delta \doteq \Delta^T$, antipode S^{-1} and counit as before. According to Drinfeld [1] a Hopf algebra A is called quasitriangular if an invertible element $R \in A \otimes A$ exists such that

$$\begin{aligned} &\Delta^T(a)R = R\Delta(a) \qquad \forall a \in A \\ &R_{13}R_{23} = (\Delta \otimes I)R \qquad R_{13}R_{12} = (I \otimes \Delta)R \end{aligned} \tag{8}$$

with the usual meaning of R_{12}, R_{13}, R_{23} as embeddings of R in $A \otimes A \otimes A$. The inverse R^{-1} is then given by $R^{-1} = (S \otimes I)R$ and it is easily shown that R satisfies the Yang–Baxter equation, $R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$.

Before introducing the algebra $B_\zeta(\alpha, \beta)$, recall that the boson algebra B is generated by a, $a^\dagger$, and N subject to the following relations:

$$[a, a^\dagger] = I \qquad [N, a] = -a \qquad [N, a^\dagger] = a^\dagger \tag{9}$$

and a Fock space representation is provided by

$$|n\rangle = \frac{1}{\sqrt{n!}}(a^\dagger)^n|0\rangle \qquad N|n\rangle = n|n\rangle$$
$$a|n\rangle = \sqrt{n}|n-1\rangle \qquad a^\dagger|n\rangle = \sqrt{(n+1)}|n+1\rangle. \tag{10}$$

The popular identification $a^\dagger a = N$ and $aa^\dagger = N+1$ holds in the quotient $B/\langle C\rangle$ (and on the above Fock space) where $\langle C\rangle$ is the two-sided ideal generated by $C = a^\dagger a - N$. As was demonstrated in [40], a Hopf algebra structure on this algebra fails to exist.

We shall now consider the family of algebras $B_\zeta(\alpha,\beta)$ generated by a, $a^\dagger$ and N subject to the following relations:

$$\begin{aligned} &aa^\dagger - \zeta a^\dagger a = \alpha N + \beta I \qquad \zeta = \pm 1 \\ &[N, a] = -a \\ &[N, a^\dagger] = a^\dagger \end{aligned} \tag{11}$$

where $\alpha, \beta \in \mathbb{R}$. If we take the quotient of $B_{-1}(2, 1)$ with respect to an ideal generated by $a^\dagger a - N$ we recover $B/\langle C\rangle$ above. Although a Hopf algebra structure for $B_1(\alpha,\beta)$ exists (see (13) below), $B_{-1}(\alpha,\beta)$ has to be enlarged to become a Hopf algebra by adding an invertible element $(-1)^N$ which will be treated as a supplementary generator satisfying the following relations:

$$\{(-1)^N, a\} = 0 = \{(-1)^N, a^\dagger\} \qquad [(-1)^N, N] = 0 \tag{12}$$

where hereafter $\{x, y\} = xy + yx$. Similar considerations were used in [46] and in that paper's context our enlarged algebra $B_{-1}(\alpha,\beta)$ can be thought of as a spectrum-generating algebra for the ordinary harmonic oscillator, while the element g of [46] will be $g = (-1)^{\tilde N}$ provided that we impose the condition $g^2 = (-1)^{2\tilde N} = I$ where $\tilde N = N + \frac{\beta}{\alpha}$. At this point we do not necessarily have to impose this condition (which implies that $(-1)^{2N} = (-1)^{-2\beta/\alpha}$). We shall denote this enlarged algebra and its universal enveloping algebras by $B^+_{-1}(\alpha,\beta)$ and $U(B^+_{-1}(\alpha,\beta))$ respectively. The coproduct, counit and antipode of $B_\zeta(\alpha,\beta)$ satisfing (7) are given by:

$$\begin{aligned} &\Delta(N) = N\otimes I + I\otimes N + \frac{\beta}{\alpha} I\otimes I \\ &\Delta(a) = a\otimes I + \zeta^{\tilde N}\otimes a \\ &\Delta(a^\dagger) = a^\dagger\otimes I + \zeta^{-\tilde N}\otimes a^\dagger \\ &\varepsilon(N) = -\frac{\beta}{\alpha} \qquad \varepsilon(a) = \varepsilon(a^\dagger) = 0 \qquad \varepsilon(I) = 1 \\ &S(N) = -N - \frac{2\beta}{\alpha} \qquad S(a) = -\zeta^{-\tilde N} a \qquad S(a^\dagger) = -a^\dagger \zeta^{\tilde N + 1} \end{aligned} \tag{13}$$

provided that $\alpha \neq 0$, together with

$$\Delta((-1)^{\pm\tilde N}) = (-1)^{\pm\tilde N}\otimes(-1)^{\pm\tilde N} \qquad \varepsilon((-1)^{\pm\tilde N}) = I \qquad S((-1)^{\pm\tilde N}) = (-1)^{\mp\tilde N}. \tag{14}$$

for the $\zeta = -1$ case. Moreover, an opposite Hopf algebra structure also exists for $B_\zeta(\alpha,\beta)$ with coproduct Δ^T and antipode the inverse $S^{-1}$of S which can be immediately deduced from S given in (13) and (14).

A Fock-type representation $B_\zeta(\alpha,\beta)$, with $a|0\rangle = 0$, $N|n\rangle = n|n\rangle, n \in \mathbb{Z}_+$ and $\langle 0|0\rangle = 1$, exists such that, when $\alpha > 0$, $\beta > 0$ is unitary and is provided by:

$$|n\rangle = \frac{1}{([n]_\zeta!)^{\frac{1}{2}}}(a^\dagger)^n|0\rangle$$

$$a|n\rangle = [n]_\zeta^{\frac{1}{2}}|n-1\rangle \qquad a^\dagger|n\rangle = [n+1]_\zeta^{\frac{1}{2}}|n+1\rangle$$

where

$$[n]_\zeta = \left(\frac{\alpha n}{2} + \frac{2\beta-\alpha}{4}(1+\zeta^{n+1})\right)\left(\frac{n(\zeta+1)+1-\zeta}{2}\right)$$
$$[n]_\zeta! = \prod_{l=1}^{n}[l]_\zeta \qquad \text{and} \qquad \langle n|n'\rangle = \delta_{nn'}. \tag{15}$$

With the definition $(-1)^{\pm N}|n\rangle = (-1)^{\pm n}|n\rangle$ this Fock space also provides a representation of $B^+_{-1}(\alpha,\beta)$. Next, we shall focus our attention mostly on certain interesting properties of $B_{-1}(\alpha,\beta)$(and $B^+_{-1}(\alpha,\beta)$).

An element L exists in the enveloping algebra $U(B_{-1}(\alpha,\beta))$ of $B_{-1}(\alpha,\beta)$ given by

$$L = \lambda_1 a^\dagger a + \lambda_2 N + \lambda_3 I \tag{16}$$

and such that

$$\{L, a\} = \{L, a^\dagger\} = 0 \tag{17}$$

provided that the following constraints on $\lambda_i \in \mathbb{R}$ $(i = 1, 2, 3)$ are satisfied

$$2\lambda_3 + \beta\lambda_1 + \lambda_2 = 0 \qquad 2\lambda_2 + \alpha\lambda_1 = 0 \tag{18}$$

(these will give for example $\lambda_2/\lambda_1 = -\alpha/2$, and $\lambda_3/\lambda_1 = (\alpha-2\beta)/4 = -\alpha/2\ (\lambda_3/\lambda_2)$, with $\alpha \neq 0$). This choice of L subject to (18) is obviously not unique as it can be easily checked that any odd power of L will satisfy (17). However, (16) is the *unique* element of a linear combination of lowest-order monomials of generators of $B_{-1}(\alpha,\beta)$ that will satisfy (17). This can be inferred by writing $L' = C_{lmn}N^l(a^\dagger)^m a^n$, $l, m, n \in \mathbb{Z}_+$, $C_{lmn} \in \mathbb{R}$ and demanding that (17) are satisfied together with $[L', N] = 0$. For a given $B_{-1}(\alpha,\beta)$, i.e. for given values of α and β relations (18) give us the conditions on λ_i under which L becomes zero. In the following we shall assume, unless otherwise stated, that L is non-zero (for example when $\alpha = 0$ and $\beta = 0$ then $L \neq 0$ if and only if $\lambda_1 \neq 0$ or when $\alpha = 0$ and $\beta \neq 0$ then $L \neq 0$ if and only if $\beta\lambda_1 = -2\lambda_3 \neq 0$). Then using (18), $L \in U(B_{-1}(\alpha,\beta))$ can be put in the form

$$L = \lambda_1\left(a^\dagger a - \frac{\alpha}{2}N + \left(\frac{\alpha}{4} - \frac{\beta}{2}\right)I\right) \qquad \lambda_1 \neq 0. \tag{19}$$

If we consider $B^+_{-1}(\alpha,\beta)$ then the additional term $\lambda_4(-1)^{\tilde{N}}$ $(\lambda_4 \in \mathbb{R}, \lambda_4 \neq 0)$ can be considered and an element $L^+ \in U(B^+_{-1}(\alpha,\beta))$ can be taken as

$$L^+ = L + \lambda_4(-1)^{\tilde{N}} \tag{20}$$

satisfying (17), while for given values of α and β, L^+ is also not unique and odd powers of it will give (17). However, it should be noted that in the case of $B^+_{-1}(\alpha,\beta)$ the element $\lambda_4(-1)^{\tilde{N}}$ is the unique non-zero lowest-order monomial satisfying (17). This again can be inferred by writing $L'^+ = C_{plmn}(-1)^{pN}N^l(a^\dagger)^m a^n$, $p, l, m, n \in \mathbb{Z}_+$, $C_{plmn} \in \mathbb{R}$ and demanding that (17) are satisfied together with $[L'^+, N] = 0$. Relation (20) is then the next most general one to be considered. Utilizing (13) and (20), $\Delta(L^+)$, $S(L^+)$ and $\varepsilon(L^+)$ can easily be found. Relations (17) are also preserved by the Hopf algebra structure (13), subject to (18), while L^+ is represented on the Fock space (15) as

$$L^+|n\rangle = \left(\lambda_1\left(\frac{\alpha}{4} - \frac{\beta}{2}\right) + \lambda_4(-1)^{\frac{\beta}{\alpha}}\right)(-1)^n|n\rangle. \tag{21}$$

Note that L^+ (and L) introduces a $\mathbb{Z}_2$ grading on the Fock space.

As mentioned, constraints (18) can be widely exploited leading to various choices of values for λ_i in terms of α, β. In the case $\alpha = 2$ and $\beta = 1$, $\lambda_1 = -\lambda_2$, $\lambda_3 = 0$ and $L^+ = \lambda_1(a^\dagger a - N) + \lambda_4(-1)^{\tilde{N}}$. Then (15) show that on this Fock space $a^\dagger a = N$, $aa^\dagger = N + I$ and $B^+_{-1}(2, 1)$ reduces to the quotient $B/\langle C\rangle$ (see the beginning of this section) extended with the element $(-1)^{a^\dagger a+1/2}$. Also we can investigate the case where we impose on $B_{-1}(\alpha, \beta)$ (or $B^+_{-1}(\alpha, \beta)$) the additional relation

$$L^2 = \eta I \qquad (\text{or } (L^+)^2 = \eta I) \tag{22}$$

with $\eta \in \mathbb{R}, \eta \neq 0$. From the form of L (see (19)) it can be easily observed that L^2 commutes with all the generators of $B_{-1}(\alpha, \beta)$ (and $B^+_{-1}(\alpha, \beta)$) and thus on any faithful representation it reduces to a multiple of the identity. Also it can be shown that (22) does not respect the Hopf algebra structure. In a representation independent way, by using (19) we obtain the characteristic identity for $C = a^\dagger a - \frac{\alpha}{2} N$

$$C\left(C + \left(\frac{\alpha}{2} - \beta\right) I\right) + \left(\frac{\alpha}{4} - \frac{\beta}{2}\right)^2 I = \frac{\eta}{\lambda_1^2} I \tag{23}$$

which when solved will give L as a multiple of the identity. Obviously, (22) with the choice of L given by (19), *is not* compatible with relations (17). The same incompatibility is also true for the case of $B^+_{-1}(\alpha, \beta)$ and L^+ given in (20). However if we consider the element $\lambda_4(-1)^{\tilde{N}} = L^+$ alone (thus letting $\lambda_1 = 0$) then (22) can hold (it is just imposing the requirement $g^2 = I$) and the Hopf algebra is preserved.

Finally, it is important in section 3, to observe that if we substitute into (11) the generators N obtained from (19) (or (20)) as $N = \frac{2}{\alpha}(-\frac{1}{\lambda_1} L + a^\dagger a + \frac{\alpha}{4} - \frac{\beta}{2})$, then (11) becomes

$$[a, a^\dagger] = -\frac{2}{\lambda_1} L + \frac{\alpha}{2} I. \tag{24}$$

This is true if and only if the values of α and β are such that L and L^+ contain the monomial N, that is when the following values of the pair (α, β) *are not* considered: $\alpha = 0, \beta = 0$ and $\alpha = 0, \beta \neq 0$. Relation (24) shows the potentiality of $B_{-1}(\alpha, \beta)$ (and $B^+_{-1}(\alpha, \beta)$) to accommodate and interchange both commutation and anticommutation relations. It is interesting to investigate whether this relation together with (17) serve as an alternative definition of $B_{-1}(\alpha, \beta)$ (or $B^+_{-1}(\alpha, \beta)$). This will become clearer in section 3 where (24) and (17) will be compared with (40).

From $B_{-1}(\alpha, \beta)$ we can obtain a realization of $B(0/1) \simeq \mathrm{osp}(\frac{1}{2})$ by introducing a $\mathbb{Z}_2$ grading such that a and $a^\dagger$ are odd and N is even and defining

$$e = \mu a^\dagger \qquad f = \lambda a \qquad h = 2N + \frac{2\beta}{\alpha} I \tag{25}$$

provided $\alpha \neq 0$ and $\mu\lambda = \frac{2}{\alpha}$, so that

$$\{e, f\} = h \qquad [h, e] = 2e \qquad [h, f] = -2f. \tag{26}$$

Then the Hopf algebra structure of $B^+_{-1}(\alpha, \beta)$ induces a non-trivial one for $\mathrm{osp}(\frac{1}{2})$ extended by the element $g = (-1)^{\tilde{N}}$, exactly as in the case of [46] (but using $B_{-1}(\alpha, \beta)$ instead of the ordinary oscillator algebra) and the R-matrix is given by (39) below. The Casimir invariant $I_2 = -\frac{1}{4}e^2 f^2 - \frac{1}{4}ef + \frac{1}{16}h^2 - \frac{1}{8}h$ on the Fock space (15) takes the eigenvalue

$$i_2 = \frac{\beta^2}{4\alpha^2} - \frac{\beta}{4\alpha} \tag{27}$$

which shows that the representation is irreducible. Similarly we can obtain a realization of A_1 (as a subalgebra of $B(0/1)$ for example) by defining

$$e' = \mu' e^2 \qquad f' = \lambda' f^2 \qquad h' = \tfrac{1}{2} h \tag{28}$$

provided $\mu'\lambda' = -\frac{1}{4}$, so that

$$[e', f'] = h' \qquad [h', e'] = 2e' \qquad [h', f'] = -2f'.$$

We can also obtain a realization of $\mathrm{sl}(2, R) \simeq \mathrm{su}(1, 1)$ if we set $J_0 = \frac{1}{2}h'$, $J_+ = \frac{\mathrm{i}}{\sqrt{2}}e'$ and $J_- = \frac{\mathrm{i}}{\sqrt{2}}f'$. Then the eigenvalues of the $\mathrm{sl}(2, R)$ Casimir invariant $C_2 = 2J_-J_+ - J_0^2 - J_0$ on the Fock space (15) are given by

$$c_n = \frac{1}{2} - \frac{\beta}{2\alpha} - \frac{\beta^2}{4\alpha^2} + \frac{2\beta - \alpha}{8\alpha}(3 + (-1)^n) \tag{29}$$

which shows that the representation is completely reducible with the two invariant subspaces corresponding to n being even and n being odd. The Casimir eigenvalues c_{even} and c_{odd} are

$$c_{\mathrm{even}} = -\frac{1}{4}\frac{\beta^2}{\alpha^2} + \frac{\beta}{2\alpha} \qquad c_{\mathrm{odd}} = \frac{1}{4} - \frac{\beta^2}{4\alpha^2}.$$

From the algebra $B_1(\alpha, \beta)$, an A_1 realization can be obtained by setting

$$e = \xi a^\dagger \qquad f = \vartheta a \qquad h = 2N + \frac{2\beta}{\alpha} I \tag{30}$$

where $\xi\vartheta = -2/\alpha$ and with the defining relations of A_1 as shown below (28).

Finally, it should be noted that an R-matrix will turn out to be trivial when $\zeta = 1$ or given by (39) when $\zeta = -1$, as it will be demonstrated in the next section.

3. Deformed boson algebras $B^q_\zeta(\alpha, \beta)$

We turn now to a q-deformation (q generic) of the algebra $B_\zeta(\alpha, \beta)$. Define $B^q_\zeta(\alpha, \beta)$ as the Lie algebra generated by a_q, $a_q^\dagger$ and N subject to the following relations:

$$\begin{aligned} &a_q a_q^\dagger - \zeta a_q^\dagger a_q = [\alpha N + \beta]_q \qquad \zeta = \pm 1 \\ &[N, a_q] = -a_q \\ &[N, a_q^\dagger] = a_q^\dagger \end{aligned} \tag{31}$$

where $\alpha, \beta \in \mathbb{R}$ and $[x]_q = (q^x - q^{-x})/(q - q^{-1})$. This algebra is a Hopf algebra whose coproduct, counit and antipode satisfy (7) and are given by:

$$\begin{aligned} &\Delta(N) = N \otimes I + I \otimes N + \frac{\beta}{\alpha} I \otimes I \\ &\Delta(a) = a_q \otimes q^{\frac{\alpha\tilde{N}}{2}} + \zeta^{\tilde{N}} q^{-\frac{\alpha\tilde{N}}{2}} \otimes a_q \\ &\Delta(a_q^\dagger) = a_q^\dagger \otimes q^{\frac{\alpha\tilde{N}}{2}} + \zeta^{-\tilde{N}} q^{-\frac{\alpha\tilde{N}}{2}} \otimes a_q^\dagger \\ &\varepsilon(N) = -\frac{\beta}{\alpha} \qquad \varepsilon(a_q) = \varepsilon(a_q^\dagger) = 0 \qquad \varepsilon(I) = 1 \\ &S(N) = -N - \frac{2\beta}{\alpha} \qquad S(a_q) = -\zeta^{-\tilde{N}} q^{-\frac{\alpha}{2}} a \qquad S(a_q^\dagger) = -a_q^\dagger \zeta^{\tilde{N}+1} q^{\frac{\alpha}{2}} \end{aligned} \tag{32}$$

provided that $\alpha \neq 0$. An opposite Hopf algebra structure also exists with coproduct Δ^T and antipode the inverse $S^{-1}$of S, which can be immediately deduced from S given in (32). Similarly to the undeformed case, in order to obtain a Hopf algebra structure for $B^q_{-1}(\alpha, \beta)$, we have to enlarge it by adding an invertible element $(-1)^{\tilde{N}}$ which will be treated as a supplementary generator satisfying relations (12) (with a_q and $a_q^\dagger$ in the place of a and $a^\dagger$ respectively) and (14). We shall denote this extended algebra (its universal enveloping algebra) as $B_{-1}^{q+}(\alpha,\beta)$ $(U(B_{-1}^{q+}(\alpha,\beta)))$.

A Fock-type representation of $B_\zeta^q(\alpha,\beta)$, with $a_q|0\rangle_q = 0$, $N|n\rangle_q = n|n\rangle_q$, $n \in \mathbb{Z}_+$ and ${}_q\langle 0|0\rangle_q = 1$, exists such that with $\alpha \neq 0$, $\beta \neq 0$

$$|n\rangle_q = \frac{1}{\sqrt{(n)_\zeta^q!}}(a_q^\dagger)^n|0\rangle_q$$

$$a_q|n\rangle_q = \sqrt{(n)_\zeta^q}|n-1\rangle_q \qquad a_q^\dagger|n\rangle_q = \sqrt{(n+1)_\zeta^q}|n+1\rangle_q$$

where

$$(n)_\zeta^q = \frac{1}{q-q^{-1}}\left(q^{\frac{\alpha(n-1)}{2}+\beta}\frac{(q^{\frac{\alpha n}{2}}-\zeta^n q^{-\frac{\alpha n}{2}})}{(q^{\frac{\alpha}{2}}-\zeta q^{-\frac{\alpha}{2}})} - q^{-\frac{\alpha(n-1)}{2}-\beta}\frac{(q^{-\frac{\alpha n}{2}}-\zeta^n q^{\frac{\alpha n}{2}})}{(q^{-\frac{\alpha}{2}}-\zeta q^{\frac{\alpha}{2}})}\right)$$
$$(n)_\zeta^q! = \prod_{m=1}^{n}(m)_\zeta^q \qquad {}_q\langle n|n'\rangle_q = \delta_{nn'}. \tag{33}$$

With the definition $(-1)^{\pm N}|n\rangle = (-1)^{\pm n}|n\rangle$ this Fock space also provides a representation of $B_{-1}^{q+}(\alpha,\beta)$. In the limit $q \to 1$ we get the Fock space of the undeformed algebra $B_\zeta(\alpha,\beta)$ (and $B_{-1}^{+}(\alpha,\beta)$).

$B_{-1}(\alpha,\beta)$ provides us with a realization of $\mathrm{osp}_{q'}(\frac{1}{2})$, with $q' = q^\alpha$, which can be obtained by defining

$$e = \mu a_q^\dagger \qquad f = \lambda a_q \qquad h = N + \frac{\beta}{\alpha}I \tag{34}$$

so that with $\mu\lambda = [\alpha]_q^{-1}$ the following $\mathrm{osp}_{q'}(\frac{1}{2})$ defining relations are satisfied:

$$\{e, f\} = [h]_{q'} \qquad [h, e] = e \qquad [h, f] = -f \tag{35}$$

while $B_1(\alpha,\beta)$ provides a $\mathrm{sl}_{q^{\alpha/2}}(2)$ realization by identifying

$$e = \xi a_q^\dagger \qquad f = \vartheta a_q \qquad h = 2N + \frac{2\beta}{\alpha}I \tag{36}$$

where $\xi\vartheta = -[\frac{\alpha}{2}]_q$, so that the defining $\mathrm{sl}_{q^{\alpha/2}}(2)$ relations below are satisfied:

$$[e, f] = [h]_{q^{\sigma/2}} \qquad [h, e] = 2e \qquad [h, f] = -2f. \tag{37}$$

Finally, an $\boldsymbol{R}$-matrix for $B_1^q(\alpha,\beta)$ and $B_{-1}^{q+}(\alpha,\beta)$ exists and is given by

$$R = R_0 q^{\alpha\tilde{N}\otimes\tilde{N}}\sum_{l=0}^{\infty}(q-q^{-1})^l q^{\zeta\frac{\alpha}{4}l(l+1)}\frac{(-\zeta)^l\zeta^{\frac{1}{4}l(l-1)}}{[l]_x!}q^{\frac{\alpha}{2}l\tilde{N}}\zeta^{l\tilde{N}}(a_q^\dagger)^l \otimes q^{-\frac{\alpha}{2}l\tilde{N}}a_q^l \tag{38}$$

where $\tilde{N} = N + \beta/\alpha$, $x = (\zeta q^{\zeta\alpha})^{1/2}$ and

$$R_0 = \tfrac{1}{2}(I\otimes I + I\otimes\zeta^{\tilde{N}} + \zeta^{\tilde{N}}\otimes I - \zeta^{\tilde{N}}\otimes\zeta^{\tilde{N}}) \tag{39}$$

provided that when $\zeta = -1$, we demand that $(-1)^{2\tilde{N}} = I$. Equation (38) has been calculated using quantum double techniques similar to [44]. It is important to mention that (38) for $\zeta = -1$, is exactly the same as the one in relation (49) of [46] with $q \to q^\alpha$, provided we do the following identifications with the generators of the bosonization of $\mathrm{osp}_{q^\alpha}(\frac{1}{2})$: $J_z = \frac{1}{2}\tilde{N}$, $V_+ = ka_q^\dagger$, $V_- = ta_q$, $g = (-1)^{\tilde{N}}$, and $kt = -[4]_{q^\alpha}^{-1}[\alpha]_q^{-1}$. Then we can argue that $B_{-1}^{q+}(\alpha,\beta)$ is the spectrum generating quantum group for the ordinary q-deformed harmonic oscillator defined by the relations $a_q a_q^\dagger - q^{\pm 1} a_q^\dagger a_q = q^{\mp N}$ and the last two of (31). The R-matrices for the undeformed Hopf algebras of section 2 can now be read off from (38) and (39) at the limit $q \to 1$, where $R \to R_0$ which for $B_1(\alpha, \beta)$ is just the identity.

4. The generalized 'ν-deformed' Heisenberg algebra $H_{\delta,\nu}$

We shall now generalize the so-called 'deformed' Heisenberg algebra of Vassiliev [12]. This is defined as the algebra $H_{\delta,\nu}$ generated by b, $b^\dagger$ and K subject to the following relations:

$$\begin{aligned} &[b, b^\dagger] = \delta I + \nu K \\ &\{K, b\} = \{K, b^\dagger\} = 0 \end{aligned} \tag{40}$$

where $\delta, \nu \in \mathbb{R}$. If we impose the additional requirements that $K^2 = I$ then with $\delta = 1$ we obtain that of [12], used for example in [13, 29, 30].

A Fock-type representation (a generalization of that appearing in [29, 30]), with $b|0>=$ 0 and $\langle 0|0\rangle = 1$, exists so that if $\nu > e0$ and $\delta > 0$ a unitary representation of $H_{\delta,\nu}$ is provided by:

$$\begin{aligned} &|m\rangle = \frac{1}{([m]!)^{\frac{1}{2}}} (b^\dagger)^m |0\rangle \\ &b|m\rangle = [m]^{\frac{1}{2}} |m-1\rangle \qquad b^\dagger|m\rangle = [m+1]^{\frac{1}{2}} |m+1\rangle \\ &K|m\rangle = \frac{\nu - \delta + 1}{\nu} (-1)^m |m\rangle \end{aligned}$$

where

$$\begin{aligned} &[m] = \delta m + \frac{\nu - \delta + 1}{2} (1 + (-1)^{m+1}) \\ &[m]! = \prod_{l=1}^{m} [l] \qquad \langle m|m'\rangle = \delta_{mm'} \end{aligned} \tag{41}$$

$m \in \mathbb{Z}_+$. The striking similarity of the Fock spaces (15) for $\zeta = -1$ and (41) is not accidental. As we shall just show under certain conditions we can obtain $B_{-1}(\alpha, \beta)$ from $H_{\delta,\nu}$ and vice versa, not only on the above Fock spaces but as abstract algebras. $H_{\delta,\nu}$ can be extended so that the resulting algebra will possess a Hopf algebra structure. There exists in the enveloping algebra $U(H_{\delta,\nu})$ of $H_{\delta,\nu}$ an element M given by

$$M = \mu_1 b^\dagger b + \mu_2 K + \rho I \tag{42}$$

and satisfying

$$[M, b] = -b \qquad [M, b^\dagger] = b^\dagger \tag{43}$$

where $\mu_i, \rho \in \mathbb{R}$ $(i = 1, 2)$ provided that the following constraint is satisfied

$$\mu_1 \delta I + (2\mu_2 - \nu\mu_1) K = I. \tag{44}$$

This suggests that, since K *should not* be a multiple of the identity, (as this contradicts the second equation of (40)) necessarily $\delta \neq 0$ which leads to $\mu_1 = 1/\delta$ and $2\mu_2 - \nu\mu_1 = 0$. Consequently with the above constraints (42) now becomes

$$M = \frac{1}{\delta} b^\dagger b + \frac{\nu}{2\delta} K + \rho I \tag{45}$$

and

$$M|m\rangle = \left(m + \frac{\nu - \delta + 1}{2\delta} + \rho \right) |m\rangle \tag{46}$$

which by choosing $\rho = -\frac{\nu-\delta+1}{2\delta}$, $M|m\rangle = m|m\rangle$. The choice of M given by (45) is obviously not the most general possible but it is the unique non-zero combination of lowest-order monomials of generators of $H_{\delta,\nu}$ that satisfy (43) while such an element does not exist if $\delta = 0$ (these considerations can be inferred by writing $M' = C_{lmn}K^l(b^\dagger)^m b^n$, $l, m, n \in \mathbb{Z}_+$, $C_{lmn} \in \mathbb{R}$, and demanding that (43) are satisfied together with $[M', K] = 0$). Now we are in a position to demonstrate the similarities between $B_{-1}(\alpha, \beta)$ and $H_{\delta,\nu}$. Solving (45) with respect to K and substituting into the first equation of (40) we obtain

$$\{b, b^\dagger\} = 2\delta M + \delta(1 - 2\rho)I \tag{47}$$

so that together with (43) $H_{\delta,\nu}$ takes the form of the defining relations of $B_{-1}(\alpha, \beta)$, (11) by setting

$$\alpha = 2\delta \qquad \beta = \delta - 2\rho\delta \tag{48}$$

and where M is replaced by N. Note that for the case where we choose $\rho = -\frac{\nu-\delta+1}{2\delta}$, $H_{\delta,\nu}$ takes the form of $B_{-1}(2\delta, \nu + 1)$. This process can also be carried out in the opposite direction, as (17) and (24) suggest, by setting in (24)

$$\delta = \alpha/2 \qquad \nu = -2/\lambda_1 \tag{49}$$

and where L is replaced by K. It is easy to observe that, as $\lambda_1 \neq 0$, (49) shows that $B_{-1}(\alpha, \beta)$ cannot be mapped to a $H_{\delta,0}$-form. Also $B_{-1}(0, \beta)$ cannot be mapped to a $H_{\delta,\nu}$-form at all, since the appropriate L fails to exist (no monomial N is present in L even if we perform a thorough search for a more general L in $U(B_{-1}(\alpha, \beta))$). Also (48) shows that a $B_{-1}(\alpha, \beta)$-form of $H_{0,\nu}$ fails to exist since M cannot be defined and for $H_{\delta,0}$ the appropriate M does not exist (no monomial K is present in M) thus also not allowing a $B_{-1}(\alpha, \beta)$-form. Consequently provided that we keep away from the values $\alpha = \delta = \nu = 0$ we can always obtain a $H_{\delta,\nu}$-form of $B_{-1}(\alpha, \beta)$ and vice versa. Relations (48) and (49) also imply that ρ and β can have arbitrary values. However, an observation of the Fock spaces (15) and (41) and a comparison of the action of K, M, L and N on them, shows that with identifications (48), (49), $\rho = -\frac{\nu-\delta+1}{2\delta}$ and $\beta = \nu + 1$ not only are these spaces equivalent but also $H_{\delta,\nu}$ and $B_{-1}(\alpha, \beta)$ are isomorphic with $K \equiv L$, $N \equiv M$, $b \equiv a$, $b^\dagger \equiv a^\dagger$.

It can be checked that the following maps $\varphi : B_{-1}(\alpha, \beta) \to H_{\delta,\nu}$ and $\varphi' : H_{\delta,\nu} \to B_{-1}(\alpha, \beta)$ defined by:

$$\varphi(a) = b \qquad \varphi(a^\dagger) = b^\dagger \qquad \varphi(N) = \frac{2}{\alpha}b^\dagger b + \frac{\nu}{\alpha}K + \frac{\delta - \beta}{\alpha} \qquad \alpha \neq 0 \tag{50}$$

$$\varphi'(b) = a \qquad \varphi'(b^\dagger) = a^\dagger \qquad \varphi'(K) = -\frac{2}{\nu}a^\dagger a + \frac{\alpha}{\nu}N + \frac{\beta - \delta}{\nu} \qquad \nu \neq 0 \tag{51}$$

are homomorphisms *if and only if* $\alpha = 2\delta$. Moreover, $\varphi' = \varphi^{-1}$ and φ becomes an isomorphism $H_{\delta,\nu} \simeq B_{-1}(\alpha, \beta)$ *provided* that both $\alpha \neq 0$ *and* $\nu \neq 0$. φ and φ' can be thought of as defining families of maps where each member is parametrized by α, β, ν and δ, β, ν respectively and we can formally write $\varphi \equiv \varphi_{\alpha,\beta,\nu}$ and $\varphi' \equiv \varphi'_{\delta,\nu,\beta}$. So for example $B_{-1}(2, 1)$ is mapped via $\varphi_{2,1,\nu}$ to $H_{1,\nu}$ (ν a fixed chosen number) and $H_{1,\nu}$ is mapped via $\varphi^{-1}_{2,1,\nu} = \varphi_{1,\nu,1}$ back to $B_{-1}(2, 1)$. Finally it can be checked that

$$\varphi(L) = -\frac{\lambda_1\nu}{2}K \qquad \varphi'(M) = N + \frac{\beta - \delta}{2\delta} + \rho. \tag{52}$$

With the existence of M of the form of (45) we can enlarge $H_{\delta,\nu}$ by adding the invertible element $(-1)^M$, as we did in the case of $B_{-1}(\alpha, \beta)$. We shall denote this enlarged algebra by $H^+_{\delta,\nu}$ and the relations that $(-1)^M$ has to satisfy are given by:

$$\{(-1)^{\pm M}, a\} = \{(-1)^{\pm M}, a^\dagger\} = 0 \qquad [(-1)^{\pm M}, K] = 0 \tag{53}$$

and on (41) will be represented as $(-1)^M|m\rangle = (-1)^{m+\frac{\nu-\delta+1}{2\delta}+\rho}|m\rangle$. $H^+_{\delta,\nu}$ can obviously be treated in the spirit of [46] as was done with $B^+_{-1}(\alpha,\beta)$, with the element g of [46] being $g=(-1)^{\tilde{M}}$, where $\tilde{M}=M-\rho+\frac{1}{2}$. Thus $H^+_{\delta,\nu}$ can be considered as a spectrum-generating algebra of the ordinary oscillator algebra. Moreover, the isomorphism can also be extended such that $H^+_{\delta,\nu}\simeq B^+_{-1}(\alpha,\beta)$ by defining $\varphi((-1)^{\tilde{N}})=(-1)^{\tilde{M}}$, $\varphi'((-1)^{\tilde{M}})=(-1)^{\tilde{N}}$.

Then a Hopf algebra structure for $H^+_{\delta,\nu}$ is given by:

$$\begin{aligned}
&\Delta(K)=K\otimes I+I\otimes K+\frac{\delta}{\nu}I\otimes I-\frac{2}{\nu}(-1)^{-M+\rho-\frac{1}{2}}b\otimes b^\dagger+\frac{2}{\nu}(-1)^{M-\rho+\frac{1}{2}}b^\dagger\otimes b\\
&\Delta(b)=b\otimes I+(-1)^{M-\rho+\frac{1}{2}}\otimes b\\
&\Delta(b^\dagger)=b^\dagger\otimes I+(-1)^{-M+\rho-\frac{1}{2}}\otimes b^\dagger\\
&\varepsilon(K)=-\frac{\delta}{\nu}\qquad \varepsilon(b)=\varepsilon(b^\dagger)=0\qquad \varepsilon(I)=1\\
&S(K)=K\qquad S(b)=-(-1)^{-M+\rho-\frac{1}{2}}b\qquad S(b^\dagger)=b^\dagger(-1)^{M-\rho+\frac{1}{2}}
\end{aligned}\tag{54}$$

$$\begin{aligned}
&\Delta((-1)^{\pm M})=(-1)^{\pm(\frac{1}{2}-\rho)}(-1)^{\pm M}\otimes(-1)^{\pm M}\\
&S((-1)^{\pm M})=(-1)^{\mp M\pm(2\rho-1)}\qquad \varepsilon((-1)^{\pm M})=(-1)^{\pm(\rho-\frac{1}{2})}
\end{aligned}\tag{55}$$

provided that $\nu,\delta\neq 0$. As in the case of $B^+_{-1}(\alpha,\beta)$ it is not necessary to impose at this stage the condition $(-1)^{2\tilde{M}}=I$ (which implies that $(-1)^{2M}=(-1)^{2\rho-1}$). The form of $\Delta(M)$, $\varepsilon(M)$ and $S(M)$ is given by

$$\begin{aligned}
&\Delta(M)=M\otimes I+I\otimes M+(\tfrac{1}{2}-\rho)I\otimes I\\
&\varepsilon(M)=\rho-\tfrac{1}{2}\qquad S(M)=-M+2\rho-1.
\end{aligned}\tag{56}$$

It can be checked that using φ and φ' we can show that the above-mentioned isomorphism also carries to the Hopf algebra structures of $B^+_{-1}(\alpha,\beta)$ and $H^+_{\delta,\nu}$. An opposite Hopf algebra structure also exists with an antipode, the inverse of the one given above, which can be immediately deduced from it.

Finally we can obtain a realization of $\mathrm{osp}(\frac{1}{2})$ by defining

$$e=\mu b^\dagger\qquad f=\lambda b\qquad h=\frac{\nu}{\delta}K+\frac{2}{\delta}b^\dagger b+I\tag{57}$$

provided that $\mu\lambda=\frac{1}{\delta}$, while for A_1 (as a subalgebra of $\mathrm{osp}(\frac{1}{2})$) by defining

$$e'=\mu'(b^\dagger)^2\qquad f=\lambda' b^2\qquad h'=\frac{\nu}{2\delta}K+\frac{1}{\delta}b^\dagger b+\frac{1}{2}I\tag{58}$$

provided $\mu'\lambda'=-\frac{1}{4\delta^2}$. Implementing the Hopf structure of $H_{\delta,\nu}$ we can obtain a Hopf structure for the bosonization of $\mathrm{osp}(\frac{1}{2})$ as was the case for $B^+_{-1}(\alpha,\beta)$. In particular it is expected that an R-matrix for $H_{\delta,\nu}$ will be of the form of (39) with $\tilde{M}$ in the place of $\tilde{N}$, provided we also demand that $(-1)^{2\tilde{M}}=I$.

5. Conclusion

In this paper we considered the generalized boson algebras $B_\zeta(\alpha,\beta)$, $\zeta=\pm1$, and their q-deformed versions $B^q_\zeta(\alpha,\beta)$. It was all shown to admit a quasitriangular Hopf algebra structure provided we also enlarge $B_{-1}(\alpha,\beta)$ and $B^q_{-1}(\alpha,\beta)$ by the element $(-1)^N$. In particular this structure revealed the property that $B^+_{-1}(\alpha,\beta)$ and $B^{q+}_{-1}(\alpha,\beta)$ can be treated as spectrum-generating quantum groups for the undeformed and q-deformed bosons respectively. Although $B_1(\alpha, \beta)$ and $B_1^q(\alpha, \beta)$ can be thought of as a more 'natural' generalization and q-deformation of the ordinary boson algebra B, it is the $\zeta = -1$ case that is important because of the isomorphism of $B_{-1}(\alpha, \beta)$ and $H_{\delta,\nu}$ (and their respective enlargements $B_{-1}^+(\alpha, \beta)$ and $H_{\delta,\nu}^+$) demonstrated in section 4 which carries over to their Hopf algebra structure. To our knowledge it is the first time that the Calogero–Vasiliev ν-deformed Heisenberg algebra $H_{1,\nu}$, slightly modified (i.e. $K^2 \neq I$), can be formulated as a Hopf algebra. Moreover, it is expected that there should exist a q-deformation, $H_{\delta,\nu}^q$ other than the one of [14] or [13] which may admit a Hopf structure, giving a two-parameter deformation of the Heisenberg algebra and possibly not being isomorphic with $B_{-1}^q(\alpha, \beta)$, thus giving rise to a new R-matrix. Consequences of these Hopf-type boson algebras on physical models such as the Calogero–Sutherland models, supersymmetric quantum mechanics, anyonic systems (whose references are mentioned in the introduction) or on radial problems, BRST symmetry [48], are under investigation. It is anticipated that the Hopf algebra structure, and especially the quasitriangular nature of these algebras, might reveal interesting connections with the integrability of the above physical systems.

Another important aspect of these models is their relations with existing ones. In the work under completion [45] we investigate the various quotients and subalgebras of these undeformed and deformed models using the powerful tool of the fixed point set of the adjoint action of a Hopf algebra. It is shown how known undeformed and q-deformed boson algebras appear as fixed-point subalgebras or as appropriate quotients. It is at this point that the role of the Cuntz algebra is also investigated.

Braid group representations and possible link invariants for all of the proposed models of deformed and undeformed bosons are worth investigating, while the $\mathrm{osp}(\frac{1}{2})$ and A_1 realizations obtained point towards realizations of higher rank algebras and superalgebras which will also allow the construction of families of infinite-dimensional representations when the above Fock spaces are generalized.

Finally, one should comment on the implications of the generalized boson algebras, in particular $B_{-1}(\alpha, \beta)$, and $B_{-1}^q(\alpha, \beta)$, for quantum statistics. As the usual oscillator algebra does not possess a Hopf algebra structure, it is difficult to characterize the multiparticle Hamiltonian. However, in our case by generalizing to a many-particle system $B_{-1}^i(\alpha, \beta)$ $i = 1, 2, \ldots$ (and in particular taking $\alpha = 2, \beta = 1$ which on the Fock space will give $a^{i\dagger}a^i = N^i$, $a^i a^{i\dagger} = N^i + I$) the total Hamiltonian (taken to be proportional to $\alpha\tilde{N} = \{a, a^\dagger\}$) has a very natural interpretation as being proportional to $\Delta^{(n)}(\tilde{N})$, the n-fold coproduct of the one-particle Hamiltonian. Perhaps of most interest are the implications for the quantum statistics of $B_{-1}^q(\alpha, \beta)$. In this case the existence of the coproduct of $\tilde{N}$ implies various logical possibilities for the multiparticle Hamiltonian, which may be more acceptable than the obvious (but arbitrary) choice $\propto \sum_i [N_i]_q$ which has no justification in terms of a Hopf structure. Non-local effects will probably emerge from such choices, which may play a crucial role in modifying the partition functions and statistics of the system.

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